Water Rocket Nozzle Analysis

Evaluating the importance of considering the nozzle design in water rocket performance prediction

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Abstract— **Two water rocket performance prediction models are presented and compared to the data of an experimental setup. The black box model is a widely used model in literature that only takes the outlet radius of the water rocket nozzle into consideration. The internal model accounts for the shape of the nozzle, including its dimensions. The nozzle shape is found to have a significant impact on the water rocket performance and the internal model predicts the rocket performance more accurately.**

I. INTRODUCTION

Nozzle design is a well studied subject in engineering, as nozzles are used in a wide variety of applications such as water turbines, rockets and water jets [1]–[4]. However, nozzle design in water rockets is not studied in engineering academically. The optimization of a water rocket nozzle differs from more general nozzle design because the pressure and mass of the rocket are not constant, and the optimized value is height of the rocket instead of thrust. The available literature on water rockets all uses a black box model that only considers the nozzle outlet radius and thus ignores all other nozzle parameters [5]–[7].

The nozzle optimization model described in Yang, Xie & Nie [4] is improved and adapted to water rocket nozzle design. This model shall be referred to as 'internal model' and takes the shape of the nozzle, including its dimensions, fully into consideration.

The aim of this research is to compare the black box model and the internal model with experimental data, thereby answering the question of how important nozzle design in water rocket performance prediction is. Both models are designed to find the nozzle outlet radius that results in the greatest maximum height reached by the water rocket. The two theoretical models are validated against data from an experimental setup using statistically justified validation techniques [8]–[10].

II. THEORETICAL ANALYSIS

In this section two water rocket performance prediction models are presented: the black box model, that does not consider the shape of the nozzle but only its outlet radius, and the internal model which takes the shape of the nozzle including its dimensions fully into consideration. Comparing these models with each other and with experimental data provides useful insights into the importance of considering the nozzle design in water rocket performance prediction.

The only two quantities that determine the dynamics of a rocket are the resultant force acting on the rocket, $F_{res}(t)$, and its mass, $m(t)$. Therefore, to make model predictions of water rocket performance these quantities as functions of time are required. The models differ in the way the thrust is calculated.

A. Height Determination

At each point in time, the resultant force on the rocket and its mass define the rocket's acceleration. The resultant force is assumed to be the vector sum of thrust, weight and drag. The acceleration at time t is assumed constant during a short time interval dt and is being used to approximate the height gain of the rocket during dt by using Euler forward integration. Using numerical integration, the instantaneous mass of the rocket is determined by subtracting the water loss during dt from the rocket's prior total mass. For both thrust models the pressure at the gas-liquid interface $P_i(t)$ of the rocket has to be determined. By assuming a polytropic expansion of the air during launch, which is justified by Romanelli, Bove & Madina [11], $P_i(t)$ at time t is given by:

$$P_i(t) = P_i(t_0)\left(\frac{V_{air,0}}{V_{air}(t)}\right)^{\kappa}. \quad (1)$$

Here, V_{air} is the volume of air enclosed in the rocket and $\kappa \equiv c_p/c_v$ is the ratio of specific heats of the air, whose approximate numerical value is taken to be 1.1, substantiated by Prusa [7].

B. Black Box Model

A relatively simple model that considers the nozzle as a black box was created. Here, the exit area is assumed to be the only nozzle parameter that affects the thrust of the rocket. The rocket's thrust is given by:

$$F_{thrust}(t) = \dot{m}(t)w_{ex}(t). \quad (2)$$

Where w_{ex} is the exit velocity and $\dot{m}$ the rate at which mass is ejected. The exit velocity can be approximated using Bernoulli's equation [5], [6]:

$$w_{ex}(t) = \sqrt{\frac{2(P_i(t) - P_{atm})}{\rho_w}}. \quad (3)$$

Where $P_i(t)$ is the absolute pressure at the gas-liquid interface, P_{atm} the atmospheric pressure and ρ_w the density of water. The mass flow rate $\dot{m}$ is equal to the exit velocity times the water density and the exit area of the nozzle, and is used to determine the mass of the rocket at time t using numerical integration.

C. Internal Model

The internal model is adapted and improved from Yang et al. [4] and based on solving the resultant force of the wall pressure distribution in the nozzle and the rocket with

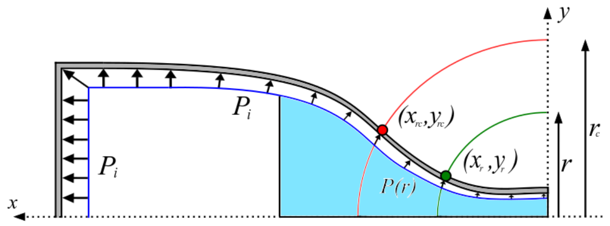

Fig. 1: A graphical representation of the coordinate system used for the internal model and the pressure distribution inside the rocket

the use of a surface integral. This force is the thrust of the rocket. In the internal model a specific coordinate is used, namely r the semi major axis length of the ellipsoid isobars in the nozzle. This coordinate is used to facilitate pressure and flow calculations. A graphical aid is provided in Fig. 1. The formula for the absolute pressure distribution in the nozzle using this coordinate is given by:

$$\begin{cases} P(r,t) = P_i(t)\left(1 - \frac{(1-P_{atm}/P_i(t))r_o^4}{(1+\delta)C_f(r)^2r^4}\right) & \text{if } r_o \leq r < r_c \\ P(r,t) = P_i(t) & \text{if } r_c \leq r \leq r_i \end{cases}. \quad (4)$$

Here r_o is the outlet radius of the nozzle in meters, δ is a minor loss factor as defined on page 345 of White [12], $C_f(r)$ is a coefficient that captures the area of the isobaric surfaces in the nozzle as in [4], r is the length of the semi-major axis of an isobar in meters and r_c is the isobar coordinate after which the pressure stops varying. Integrating Eq. 4 over the nozzle surface and adding this force vector and the resultant force of the pressure in the air chamber to find the thrust, provides the following formula:

$$F_{thrust}(t) = P_{g,i}(t)\pi r_o^2 + \frac{2P_{g,i}(t)\pi r_o^4}{(1+\delta)}\int_{r_o}^{y_{rc}} \frac{1}{C_f(r)^2r^4} y_r dy_r. \quad (5)$$

Here $P_{g,i}(t)$ is the gauge pressure at the gas-liquid interface in Pascal, y_{rc} is the y-coordinate of the intersection between the critical isobar with r_c and the nozzle wall and y_r are the y-coordinates of the intersection points of isobars with corresponding coordinates r and the nozzle wall.

D. Optimization Method

For the internal model an optimization method is used to determine the optimal exit radius. This optimization routine is required since different nozzle parameters result in different minor loss factors. The following steps should be followed:

1) Determine an initial optimal nozzle guess with $r_{o,0}$ and δ_0.
2) Solve Eq. 6 for $r_{o,i}$, where H_{max} is the maximum height reached by the rocket:

$$\left.\frac{\partial H_{max}}{\partial r_o}\right|_{\delta_i}(r_{o,i}) = 0. \quad (6)$$

3) Determine δ_i corresponding to the new $r_{o,i}$ using numerical analysis or experiments.
4) Repeat steps 2 and 3 until $r_{o,i}$ converges.

For the black box model only Eq. 6 has to be evaluated to find an optimal outlet radius.

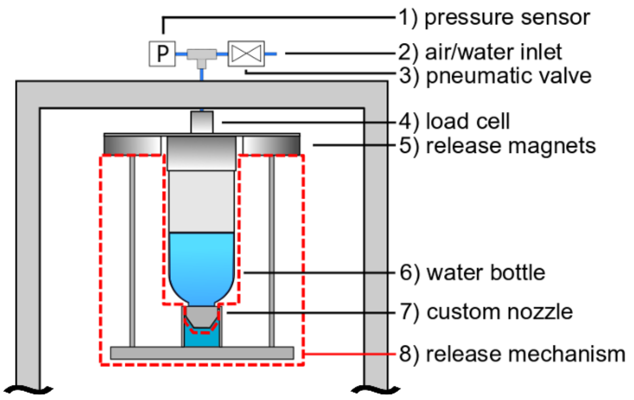

Fig. 2: Schematic view of the experimental setup

III. EXPERIMENTAL SETUP

An experimental setup is used to validate the theoretical models. Resultant force and air pressure in the rocket are the quantities measured. From the air pressure, the mass of the rocket can be deduced. Resultant force and mass as functions of time define the rocket dynamics. Comparing these quantities to the predictions of the models thus provides data for statistical validation. The water rocket is fixed to the setup to ensure repeatability.

A schematic drawing of the experimental setup is given in Fig. 2. The rocket is a 1 liter PET bottle (6) with a relatively large opening, that ensures that the natural nozzle of the bottle has as little effect as possible. Custom 3D printed nozzles (7) are attached to the bottle by using the bottle thread.

The frame is built up with a modular profile system. Air and water are pumped into the rocket using a pneumatic circuit. Once the desired pressure is reached the pneumatic valve (3) is closed. The pressure sensor (1), with a 10 bar range, records the pressure during measurements.

The bottle and release mechanism (8) are connected to a load cell (4) with a range of 111N. The load cell is mounted to the frame.

A release mechanism (8) keeps the nozzle closed during preparation. When ready, the electromagnets (5) are turned off and the part surrounded by the dotted line falls down. A rocket launch is simulated.

The load cell and pressure sensor are recorded with a DAQ device at a rate of 1000 Hz. The measured data are imported into MATLAB in order to analyze, plot and compare the data to the models.

IV. RESULTS

The predictions are compared to data acquired with the experimental setup. A statistical analysis is provided to validate the models. During the validity analysis and the optimization routine only concave nozzles are examined. This decision was made following Theobald's [3] findings, as a nozzle with the least possible energetic losses would offer the biggest chance of the black box model correctly predicting reality.

A. Shape Dependency

To investigate whether there is a significant dependence of thrust generation on nozzle shape, three nozzles with fundamentally different shapes, but the same inlet and outlet radii were tested under the same initial conditions. The following three shapes were used: a nozzle with just a hole, a conical nozzle and a concave nozzle. The resulting force profiles are presented in Fig. 3.

B. Validity

The models are validated using a multi-stage process as proposed by Finger & Naylor [8]. The first step is to assess the face validity of the model. This is done by visual inspection of the overlap of model predictions and experimental measurements. An overlap of the force measurements and models predictions is provided in Fig. 4.

The second step of the process is to verify whether the model assumptions are correct. The assumption made for the maximum height versus exit radius plot is that there is an optimum. Fig. 5 shows the exit radius on the x-axis and the maximum height on the y-axis, as predicted by both models.

The third step is to compare the system input-output transformations to corresponding model input-output transformations. This comparison is performed using statistical methods. Cohen & Cyert [9] proposed a method using regression analysis. In Fig. 6 the experimental force results for several initial pressures are plotted on the y-axis against the model force predictions for corresponding initial pressures on the x-axis as advised by Piñeiro, Perelman, Guerschman & Paruelo [10]. Alongside these, the robust regression lines for both models calculated with the MATLAB function robustfit() are presented.

A measure for the variance relative to the fit line is given by the coefficient of determination. For the internal model this value is 58% and for the black box model this is 59%. The regression coefficients for the internal model are $b_0 = 1.0060$ and $b_1 = 0.0140$ and for the black box model $b_0 = 0.8838$ and $b_1 = -0.0952$. The regression line is given by $h(x) = b_0 x + b_1$.

C. Height Prediction

Table I shows the calculated maximum heights reached by the rocket corresponding to the force data plotted in Fig. 4.

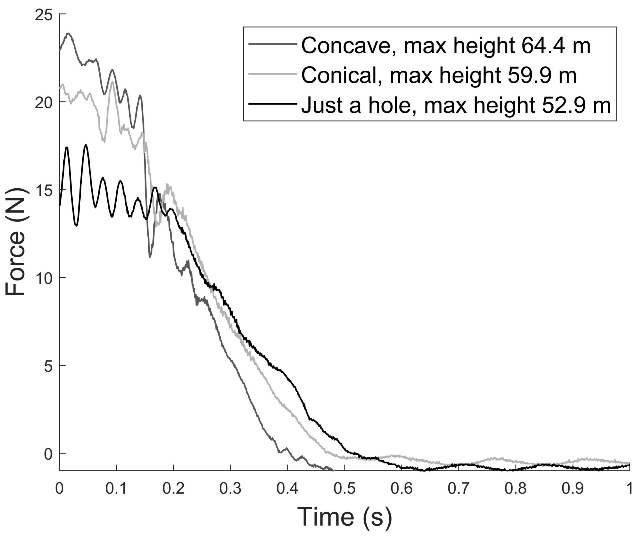

Fig. 3: A plot of the measured force as function of time for three different nozzle shapes, just a hole, conical and concave with equal initial conditions.

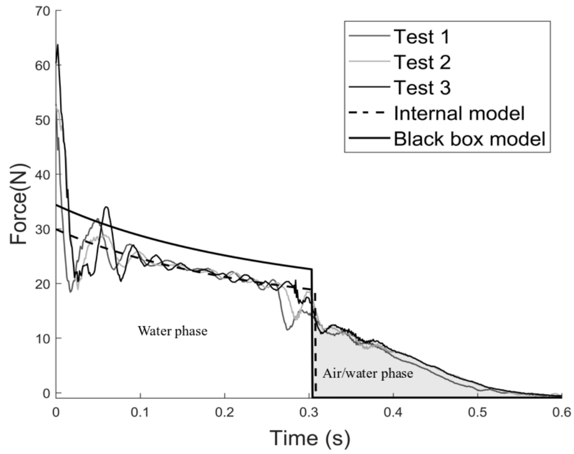

Fig. 4: A plot of the measured force as function of time for three tests with 5 bar initial conditions and equal amounts of water using a concave nozzle, with the corresponding model results displayed over them.

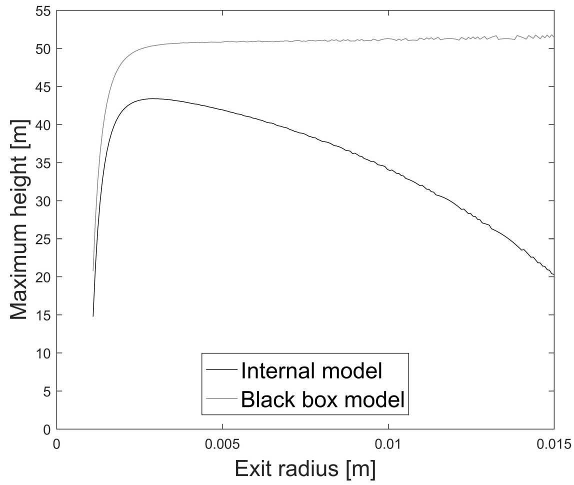

Fig. 5: Maximum reached height against exit radius for 6 bar, 1/3 water and concave nozzle shape with inlet radius of 16 mm.

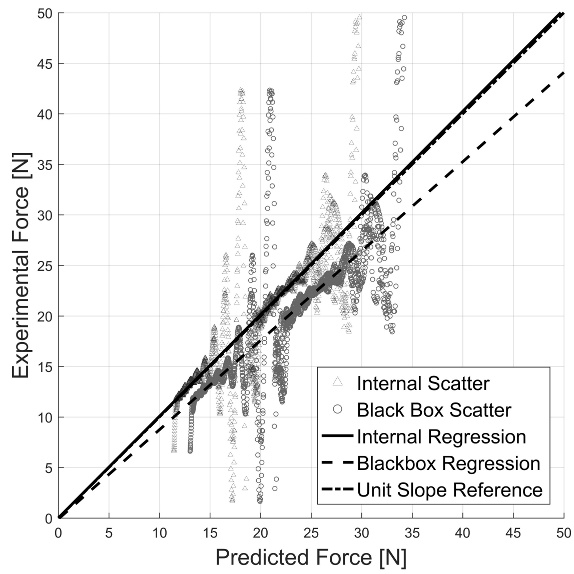

Fig. 6: A regression plot with model force on the x-axis and measured force on the y-axis, alongside a reference line with slope one that passes through the origin for nozzle tests using 4, 5 and 6 bar with a concave nozzle with an outlet radius of 3.5 mm.

'Air/water phase' refers to the last phase of the thrust that is not accounted for in the models, where the nozzle exhaust jet consists of an air/water mixture. The experimental air/water force data are added to the force predictions of the models in the height calculations shown in the rightmost column of Table I. In order to determine the significance of this thrust phase, the second column presents the model data without this extra force data.

TABLE I: Numerically calculated maximum heights reached by a predefined rocket from the data seen in Fig. 4.

	Without Air/water thrust phase	**With Air/water thrust phase**
Height test 1 [m]	-	69
Height test 2 [m]	-	69
Height test 3 [m]	-	71
Height internal [m]	38	72
Height black box [m]	46	81

D. Optimized Nozzle

The optimization routine was implemented for a predefined rocket and initial conditions of 6 bar and a water-air ratio of 1:2, using the internal model. This process converged to an optimal exit radius prediction of roughly 3.2 mm.

V. DISCUSSION

- Due to the design of the release mechanism a very high peak force is generated at the moment of release. The release mechanism can be thought of as a spring under tension which causes a force on the load cell when the release magnets are triggered. As a result of the spring characteristics of the load cell, this force results in an oscillation of the system. Only the measurements after the oscillation has mostly damped out are considered. The rocket can be seen as if it is launched with a lower pressure and water mass and can still be compared to the models. The effect of different fluid velocities inside the bottle can be neglected at any point in time, when compared to w_{ex}. Therefore the water can be assumed to be at rest at the inlet, so every point in time can be used as an approximate initial state. The new initial conditions are found using Eq. 1. In Fig. 3 and Fig. 4 these new time points are taken as $t = 0$.
- Although Thorncroft, Ridgley & Pascual [5] use a model similar to the black box model and present their results as sufficiently accurate, the internal model provides significantly better predictions. Fig. 4 shows that the internal model matches resultant forces measurements better than the black box model. This behavior can be seen in all test results. The black box model does not take into account losses in the nozzle, which explains the overestimation.
- In Fig. 6 the robust regression fit of the internal model corresponds significantly more closely to the unit slope reference than the black box model. The resulting b coefficients substantiate this. The coefficient of determination is roughly the same because the shape of both predictions is similar as can be seen in Fig. 4, hence the variance from the fit lines will be equivalent. A robust regression model was used to suppress the effect of the oscillations caused by the experimental set-up.
- Fig. 5 shows that the black box model does not predict an optimum exit radius. This result agrees with the findings of Mulsow [13]. The internal model result agrees with Mulsow's suggestion that a more sophisticated model could predict an optimum.
- When the water has almost run out, the rockets expels an air/water mixture and later pure air, this can be seen in the sudden drop in thrust. Finney [6] suggests that this portion of the thrust generation can be neglected. However, during this period, the mass of the rocket is relatively low so the impact of this force on the maximum height is significant as shown in Table I.

VI. CONCLUSION

- The nozzle shape has a significant impact on water rocket performance.
- Considering the period where only water is expelled from the rocket, the internal model is a better approximation of reality than the black box model.
- Within the range of initial conditions considered in this research, an optimal exit radius for a water rocket can be predicted when using the internal model.
- The air/water thrust phase cannot be ignored in water rocket height prediction for the initial conditions considered.

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